

PRECONDITIONING FRACTIONAL SPECTRAL COLLOCATION

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Abstract. Fractional spectral collocation (FSC) method based on fractional Lagrange interpolation has recently been proposed to solve fractional differential equations. Numerical experiments show that the linear systems in FSC become extremely ill-conditioned as the number of collocation points increases. By introducing suitable fractional Birkhoff interpolation problems, we present fractional integration preconditioning matrices for the ill-conditioned linear systems in FSC. The condition numbers of the resulting linear systems are independent of the number of collocation points. Numerical examples are given.

Key words. Fractional Lagrange interpolation, fractional Birkhoff interpolation, fractional spectral collocation, preconditioning

AMS subject classifications. 65L60, 41A05, 41A10

1. Introduction. Fractional spectral collocation (FSC) methods [7, 8, 2] based on fractional Lagrange interpolation have recently been proposed to solve fractional differential equations. By a spectral theory developed in [6] for fractional Sturm-Liouville eigenproblems, the corresponding fractional differential matrices can be obtained with ease. However, numerical experiments show that the involved linear systems become extremely ill-conditioned as the number of collocation points increases. Typically, the condition number behaves like $\mathcal{O}(N^{2\nu})$, where N is the number of collocation points and ν is the order of the leading fractional term. Efficient preconditioners are highly required when solving the linear systems by an iterative method.

Recently, Wang, Samson, and Zhao [5] proposed a well-conditioned collocation method to solve linear differential equations with various types of boundary conditions. By introducing a suitable Birkhoff interpolation problem, they constructed a pseudospectral integration preconditioning matrix, which is the exact inverse of the pseudospectral discretization matrix of the n th-order derivative operator together with n boundary conditions. Essentially, the linear system in the well-conditioned collocation method [5] is the one obtained by right preconditioning the original linear system; see [1]. By introducing suitable fractional Birkhoff interpolation problems and employing the same techniques in [5], Jiao, Wang, and Huang [3] proposed fractional integration preconditioning matrices for linear systems in fractional collocation methods base on Lagrange interpolation. In the Riemann-Liouville case, it is necessary to modify the fractional derivative operator in order to absorb singular fractional factors (see [3, §3]). In this paper, we extend the Birkhoff interpolation preconditioning techniques in [5, 3] to the fractional spectral collocation methods [7, 8, 2] based on fractional Lagrange interpolation. Unlike that in [3], there are no singular fractional factors in the Riemann-Liouville case. Numerical experiments show that the condition number of the resulting linear system is independent of the number of collocation points.

The rest of the paper is organized as follows. In §2, we review several topics required in the following sections. In §3, we introduce fractional Birkhoff interpolation

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problems and the corresponding fractional integration matrices. In §4, we present the preconditioning fractional spectral collocation method. Numerical examples are also reported. We present brief concluding remarks in §5.

2. Preliminaries.

2.1. Fractional derivatives. The definitions of fractional derivatives of order $\nu \in (n-1, n), n \in \mathbb{N}$, on the interval $[-1, 1]$ are as follows [4]:

- Left-sided Riemann-Liouville fractional derivative:

$${}^{RL}_{-1}\mathcal{D}_x^\nu u(x) = \frac{1}{\Gamma(n-\nu)} \frac{d^n}{dx^n} \int_{-1}^x \frac{u(\xi)}{(x-\xi)^{\nu-n+1}} d\xi,$$

- Right-sided Riemann-Liouville fractional derivative:

$${}^{RL}_x\mathcal{D}_1^\nu u(x) = \frac{(-1)^n}{\Gamma(n-\nu)} \frac{d^n}{dx^n} \int_x^1 \frac{u(\xi)}{(\xi-x)^{\nu-n+1}} d\xi,$$

- Left-sided Caputo fractional derivative:

$${}^C_{-1}\mathcal{D}_x^\nu u(x) = \frac{1}{\Gamma(n-\nu)} \int_{-1}^x \frac{u^{(n)}(\xi)}{(x-\xi)^{\nu-n+1}} d\xi,$$

- Right-sided Caputo fractional derivative:

$${}^C_x\mathcal{D}_1^\nu u(x) = \frac{(-1)^n}{\Gamma(n-\nu)} \int_x^1 \frac{u^{(n)}(\xi)}{(\xi-x)^{\nu-n+1}} d\xi.$$

By the definitions of fractional derivatives, we have

$$(2.1) \quad {}^{RL}_{-1}\mathcal{D}_x^\nu u(x) = \sum_{i=0}^{n-1} \frac{u^{(i)}(-1)}{\Gamma(1+i-\nu)} (x+1)^{i-\nu} + {}^C_{-1}\mathcal{D}_x^\nu u(x),$$

and

$$(2.2) \quad {}^{RL}_x\mathcal{D}_1^\nu u(x) = \sum_{i=0}^{n-1} \frac{(-1)^i u^{(i)}(1)}{\Gamma(1+i-\nu)} (1-x)^{i-\nu} + {}^C_x\mathcal{D}_1^\nu u(x).$$

Therefore,

$${}^{RL}_{-1}\mathcal{D}_x^\nu u(x) = {}^C_{-1}\mathcal{D}_x^\nu u(x), \quad \text{if } u^{(i)}(-1) = 0, \quad i = 0, 1, \dots, n-1,$$

and

$${}^{RL}_x\mathcal{D}_1^\nu u(x) = {}^C_x\mathcal{D}_1^\nu u(x), \quad \text{if } u^{(i)}(1) = 0, \quad i = 0, 1, \dots, n-1.$$

In this paper, we mainly deal with the left-sided Riemann-Liouville fractional problems with homogeneous boundary/initial conditions. By a simple change of variables, (2.1) and (2.2), the extension to other fractional problems is easy.

2.2. Fractional Lagrange interpolation. Throughout the paper, let $\{x_j\}_{j=1}^N$ be a set of distinct points satisfying

$$(2.3) \quad -1 < x_1 < \cdots < x_{N-1} < x_N \leq 1.$$

Given $\mu \in (0, 1)$, the μ -fractional Lagrange interpolation basis associated with the points $\{x_j\}_{j=1}^N$ is defined as

$$(2.4) \quad \ell_j^\mu(x) = \left(\frac{x+1}{x_j+1} \right)^\mu \prod_{n=1, n \neq j}^N \frac{x-x_n}{x_j-x_n}, \quad j = 1, \dots, N.$$

For a function $u(x)$ with $u(-1) = 0$, the μ -fractional Lagrange interpolant $u_N(x)$ of $u(x)$ takes the form

$$u_N(x) = \sum_{j=1}^N u(x_j) \ell_j^\mu(x).$$

2.3. Computations of ${}^{RL}_{-1}\mathcal{D}_x^\mu \ell_j^\mu(x)$ and ${}^{RL}_{-1}\mathcal{D}_x^{1+\mu} \ell_j^\mu(x)$ with $\mu \in (0, 1)$. Note that $\ell_j^\mu(x)$, $j = 1, \dots, N$, can be represented exactly as

$$(2.5) \quad \ell_j^\mu(x) = \left(\frac{x+1}{x_j+1} \right)^\mu \prod_{n=1, n \neq j}^N \frac{x-x_n}{x_j-x_n} = (x+1)^\mu \sum_{n=1}^N \alpha_{nj} P_{n-1}^{(-\mu, \mu)}(x),$$

where $P_n^{(\alpha, \beta)}(x)$ denote the standard Jacobi polynomials. The coefficients α_{nj} can be obtained by solving the linear system

$$\sum_{n=1}^N (x_i+1)^\mu P_{n-1}^{(-\mu, \mu)}(x_i) \alpha_{nj} = \delta_{ij}, \quad i = 1, \dots, N.$$

REMARK 2.1. Let $\{x_j\}_{j=1}^N$ and $\{\omega_j\}_{j=1}^N$ be the Gauss-Jacobi quadrature nodes and weights with the Jacobi polynomial $P_N^{(-\mu, \mu)}(x)$. Then,

$$\alpha_{nj} = \frac{1}{(x_j+1)^\mu} \frac{(2n-1)(n-1)!(n-1)!}{2\Gamma(n-\mu)\Gamma(n+\mu)} \omega_j P_{n-1}^{(-\mu, \mu)}(x_j).$$

We now compute ${}^{RL}_{-1}\mathcal{D}_x^\mu \ell_j^\mu(x)$ and ${}^{RL}_{-1}\mathcal{D}_x^{1+\mu} \ell_j^\mu(x)$. Let $P_n(x)$ denote the Legendre polynomial of order n . By (see [6])

$${}^{RL}_{-1}\mathcal{D}_x^\mu \left((x+1)^\mu P_{n-1}^{(-\mu, \mu)}(x) \right) = \frac{\Gamma(n+\mu)}{\Gamma(n)} P_{n-1}(x)$$

and

$${}^{RL}_{-1}\mathcal{D}_x^{1+\mu} \ell_j^\mu(x) = \frac{d}{dx} \left({}^{RL}_{-1}\mathcal{D}_x^\mu \ell_j^\mu(x) \right),$$

we have

$${}^{RL}_{-1}\mathcal{D}_x^\mu \ell_j^\mu(x) = \sum_{n=1}^N \alpha_{nj} \frac{\Gamma(n+\mu)}{\Gamma(n)} P_{n-1}(x)$$

and

$$\begin{aligned} {}_{-1}^{RL}\mathcal{D}_x^{1+\mu}\ell_j^\mu(x) &= \sum_{n=2}^N \alpha_{nj} \frac{\Gamma(n+\mu)}{\Gamma(n)} P'_{n-1}(x) \\ &= \sum_{n=2}^N \alpha_{nj} \frac{\Gamma(n+\mu)}{\Gamma(n)} \frac{n}{2} P_{n-2}^{(1,1)}(x). \end{aligned}$$

3. Riemann-Liouville fractional Birkhoff interpolation. Let \mathbb{P}_n be the set of all algebraic polynomials of degree at most n . Define the space

$$\mathbb{S}_N^\mu = (x+1)^\mu \mathbb{P}_{N-1}.$$

In the following, we consider two special cases.

3.1. The case $\nu = \mu \in (0, 1)$. For a function $u(x)$ with $u(-1) = 0$, given N distinct points $\{y_j\}_{j=1}^N$ satisfying

$$-1 < y_1 < \cdots < y_{N-1} < y_N \leq 1,$$

consider the Riemann-Liouville ν -fractional Birkhoff interpolation problem:

$$(3.1) \quad \text{Find } p(x) \in \mathbb{S}_N^\mu \text{ such that } {}_{-1}^{RL}\mathcal{D}_x^\nu p(y_j) = {}_{-1}^{RL}\mathcal{D}_x^\nu u(y_j), \quad j = 1, \dots, N.$$

THEOREM 3.1. *The interpolant $u_N^\nu(x)$ for the Riemann-Liouville ν -fractional Birkhoff problem (3.1) of a function $u(x)$ with $u(-1) = 0$ takes the form*

$$u_N^\nu(x) = \sum_{j=1}^N {}_{-1}^{RL}\mathcal{D}_x^\nu u(y_j) B_j^\nu(x),$$

where

$$B_j^\nu(x) = (x+1)^\nu \sum_{n=1}^N \tilde{\alpha}_{nj} P_{n-1}^{(-\nu, \nu)}(x), \quad j = 1, \dots, N,$$

with $\tilde{\alpha}_{nj}$ satisfying

$$\sum_{n=1}^N \tilde{\alpha}_{nj} \frac{\Gamma(n+\nu)}{\Gamma(n)} P_{n-1}(y_i) = \delta_{ij}, \quad i = 1, \dots, N.$$

By ${}_{-1}^{RL}\mathcal{D}_x^\nu B_j^\nu(y_i) = \delta_{ij}$, the proof of Theorem 3.1 is straightforward.

REMARK 3.2. *Let $\{y_j\}_{j=1}^N$ and $\{\omega_j\}_{j=1}^N$ be the Gauss-Legendre quadrature nodes and weights with the Legendre polynomial $P_N(x)$. Then,*

$$\tilde{\alpha}_{nj} = \frac{2n-1}{2} \frac{\Gamma(n)}{\Gamma(n+\nu)} \omega_j P_{n-1}(y_j).$$

Define the matrices

$$\mathbf{D}_{\mathbf{x} \mapsto \mathbf{y}}^{(\nu)} = [{}_{-1}^{RL}\mathcal{D}_x^\nu \ell_j^\mu(y_i)]_{i,j=1}^N, \quad \mathbf{B}_{\mathbf{y} \mapsto \mathbf{x}}^{(-\nu)} = [B_j^\nu(x_i)]_{i,j=1}^N.$$

It is easy to show that

$$(3.2) \quad \mathbf{D}_{\mathbf{x} \mapsto \mathbf{y}}^{(\nu)} \mathbf{B}_{\mathbf{y} \mapsto \mathbf{x}}^{(-\nu)} = \mathbf{I}_N.$$

3.2. The case $\nu = 1 + \mu \in (1, 2)$. For a function $u(x)$ with $u(\pm 1) = 0$, given $N - 1$ distinct points $\{y_j\}_{j=1}^{N-1}$ satisfying

$$-1 < y_1 < \cdots < y_{N-1} < 1,$$

consider the Riemann-Liouville ν -fractional Birkhoff interpolation problem:

$$(3.3) \text{ Find } p(x) \in \mathbb{S}_N^\mu \text{ such that } \begin{cases} p(1) = 0, \\ {}_{-1}^{RL}\mathcal{D}_x^\nu p(y_j) = {}_{-1}^{RL}\mathcal{D}_x^\nu u(y_j), \quad j = 1, \dots, N-1. \end{cases}$$

THEOREM 3.3. *The interpolant $u_N^\nu(x)$ for the Riemann-Liouville ν -fractional Birkhoff problem (3.3) of a function $u(x)$ with $u(\pm 1) = 0$ takes the form*

$$u_N^\nu(x) = \sum_{j=1}^{N-1} {}_{-1}^{RL}\mathcal{D}_x^\nu u(y_j) B_j^\nu(x),$$

where

$$B_j^\nu(x) = (x+1)^\mu \sum_{n=1}^{N-1} \tilde{\beta}_{nj} \left(P_n^{(-\mu, \mu)}(x) - P_n^{(-\mu, \mu)}(1) \right), \quad j = 1, \dots, N-1,$$

with $\mu = \nu - 1$ and $\tilde{\beta}_{nj}$ satisfying

$$\sum_{n=1}^{N-1} \tilde{\beta}_{nj} \frac{\Gamma(n+1+\mu)}{\Gamma(n+1)} \frac{n+1}{2} P_{n-1}^{(1,1)}(y_i) = \delta_{ij}, \quad i = 1, \dots, N-1.$$

By ${}_{-1}^{RL}\mathcal{D}_x^\nu B_j^\nu(y_i) = \delta_{ij}$, the proof of Theorem 3.3 is straightforward.

REMARK 3.4. Let $\{y_j\}_{j=1}^{N-1}$ and $\{\omega_j\}_{j=1}^N$ be the Gauss-Jacobi quadrature nodes and weights with the Jacobi polynomial $P_{N-1}^{(1,1)}(x)$. Then,

$$\tilde{\beta}_{nj} = \frac{2n+1}{4n} \frac{\Gamma(n+1)}{\Gamma(n+1+\mu)} \omega_j P_{n-1}^{(1,1)}(y_j).$$

In this subsection, let $x_N = 1$. Define the matrices

$$\tilde{\mathbf{D}}_{\mathbf{x} \mapsto \mathbf{y}}^{(\nu)} = [{}_{-1}^{RL}\mathcal{D}_x^\nu \ell_j^\nu(y_i)]_{i,j=1}^{N-1}, \quad \tilde{\mathbf{B}}_{\mathbf{y} \mapsto \mathbf{x}}^{(-\nu)} = [B_j^\nu(x_i)]_{i,j=1}^{N-1}.$$

It is easy to show that

$$(3.4) \quad \tilde{\mathbf{D}}_{\mathbf{x} \mapsto \mathbf{y}}^{(\nu)} \tilde{\mathbf{B}}_{\mathbf{y} \mapsto \mathbf{x}}^{(-\nu)} = \mathbf{I}_{N-1}.$$

4. Preconditioning fractional spectral collocation (PFSC). In this section, we use two examples to introduce the preconditioning scheme.

4.1. An initial value problem. Consider the fractional differential equation of the form

$$(4.1) \quad {}_{-1}^{RL}\mathcal{D}_x^\nu u(x) + a(x)u(x) = f(x), \quad \nu \in (0, 1); \quad u(-1) = 0.$$

The fractional spectral collocation scheme leads to the following linear system

$$(4.2) \quad \left(\mathbf{D}_{\mathbf{x} \mapsto \mathbf{y}}^{(\nu)} + \text{diag}\{\mathbf{a}\} \mathbf{D}_{\mathbf{x} \mapsto \mathbf{y}}^{(0)} \right) \mathbf{u} = \mathbf{f},$$

where

$$\mathbf{D}_{\mathbf{x} \mapsto \mathbf{y}}^{(0)} = [\ell_j^\nu(y_i)]_{i,j=1}^N,$$

and

$$\begin{aligned} \mathbf{a} &= [a(y_1) \ a(y_2) \ \cdots \ a(y_N)]^T, \\ \mathbf{f} &= [f(y_1) \ f(y_2) \ \cdots \ f(y_N)]^T. \end{aligned}$$

The unknown vector \mathbf{u} is an approximation of the vector of the exact solution $u(x)$ at the points $\{x_j\}_{j=1}^N$, i.e.,

$$[u(x_1) \ u(x_2) \ \cdots \ u(x_N)]^T.$$

Consider the matrix $\mathbf{B}_{\mathbf{y} \mapsto \mathbf{x}}^{(-\nu)}$ as a right preconditioner for the linear system (4.2). By (3.2), we have the right preconditioned linear system

$$(4.3) \quad (\mathbf{I}_N + \text{diag}\{\mathbf{a}\} \mathbf{D}_{\mathbf{x} \mapsto \mathbf{y}}^{(0)} \mathbf{B}_{\mathbf{y} \mapsto \mathbf{x}}^{(-\nu)}) \mathbf{v} = \mathbf{f}.$$

It is easy to show that

$$\mathbf{D}_{\mathbf{x} \mapsto \mathbf{y}}^{(0)} \mathbf{B}_{\mathbf{y} \mapsto \mathbf{x}}^{(-\nu)} = \mathbf{B}_{\mathbf{y} \mapsto \mathbf{y}}^{(0-\nu)},$$

where

$$\mathbf{B}_{\mathbf{y} \mapsto \mathbf{y}}^{(0-\nu)} = [B_j^\nu(y_i)]_{i,j=1}^N.$$

Then, the equation (4.3) reduces to

$$(4.4) \quad (\mathbf{I}_N + \text{diag}\{\mathbf{a}\} \mathbf{B}_{\mathbf{y} \mapsto \mathbf{y}}^{(0-\nu)}) \mathbf{v} = \mathbf{f}.$$

After solving (4.4), we obtain \mathbf{u} by $\mathbf{u} = \mathbf{B}_{\mathbf{y} \mapsto \mathbf{x}}^{(-\nu)} \mathbf{v}$.

Example 1. We consider the fractional differential equation (4.1) with

$$\nu = 0.8, \quad a(x) = 2 + \sin(25x).$$

The function $f(x)$ is chosen such that the exact solution of (4.1) is

$$u(x) = e^{x+1} - 1 + (x+1)^{46/7}.$$

Let $\{x_j\}_{j=1}^N$ be the Gauss-Jacobi points as in Remark 2.1 and $\{y_j\}_{j=1}^N$ be the Gauss-Legendre points as in Remark 3.2. We compare condition numbers, number of iterations (using BiCGSTAB in Matlab with TOL = 10^{-9}) and maximum point-wise errors of FSC and PFSC (see Figure 1). Observe from Figure 1 (left) that the condition number of FSC behaves like $\mathcal{O}(N^{1.6})$, while that of PFSC scheme remains a constant even for N up to 1024. As a result, PFSC scheme only requires about 7 iterations to converge (see Figure 1 (middle)), while the usual FSC scheme requires much more iterations with a degradation of accuracy as depicted in Figure 1 (right).

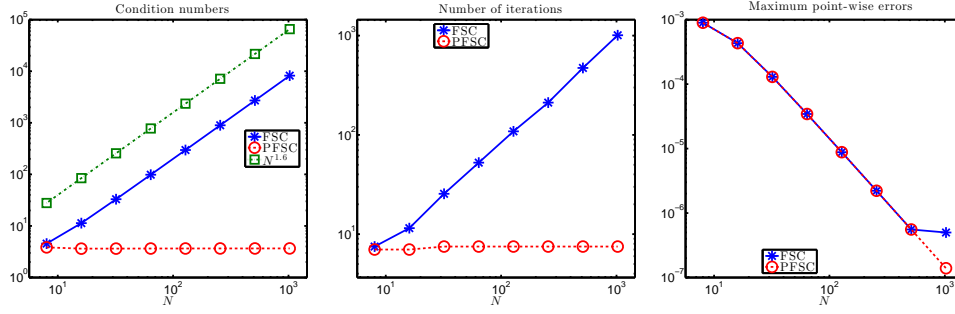


FIG. 1. Comparison of condition numbers (left), number of iterations (middle), and maximum point-wise errors (right) for Example 1.

4.2. A boundary value problem. Consider the fractional differential equation of the form

$$(4.5) \quad {}^{RL}_{-1} \mathcal{D}_x^\nu u(x) + a(x)u'(x) + b(x)u(x) = f(x), \quad \nu = 1 + \mu \in (1, 2); \quad u(\pm 1) = 0.$$

The fractional spectral collocation method leads to the following linear system

$$(4.6) \quad \left(\tilde{\mathbf{D}}_{\mathbf{x} \rightarrow \mathbf{y}}^{(\nu)} + \text{diag}\{\mathbf{a}\} \tilde{\mathbf{D}}_{\mathbf{x} \rightarrow \mathbf{y}}^{(1)} + \text{diag}\{\mathbf{b}\} \tilde{\mathbf{D}}_{\mathbf{x} \rightarrow \mathbf{y}}^{(0)} \right) \mathbf{u} = \mathbf{f},$$

where

$$\tilde{\mathbf{D}}_{\mathbf{x} \rightarrow \mathbf{y}}^{(1)} = \left[\frac{d}{dx} (\ell_j^\mu(x)) \Big|_{x=y_i} \right]_{i,j=1}^{N-1}, \quad \tilde{\mathbf{D}}_{\mathbf{x} \rightarrow \mathbf{y}}^{(0)} = [\ell_j^\mu(y_i)]_{i,j=1}^{N-1},$$

and

$$\begin{aligned} \mathbf{a} &= [a(y_1) \quad a(y_2) \quad \cdots \quad a(y_{N-1})]^T, \\ \mathbf{b} &= [b(y_1) \quad b(y_2) \quad \cdots \quad b(y_{N-1})]^T, \\ \mathbf{f} &= [f(y_1) \quad f(y_2) \quad \cdots \quad f(y_{N-1})]^T. \end{aligned}$$

The unknown vector \mathbf{u} is an approximation of the vector of the exact solution $u(x)$ at the points $\{x_j\}_{j=1}^{N-1}$, i.e.,

$$[u(x_1) \quad u(x_2) \quad \cdots \quad u(x_{N-1})]^T.$$

Consider the matrix $\tilde{\mathbf{B}}_{\mathbf{y} \rightarrow \mathbf{x}}^{(-\nu)}$ as a right preconditioner for the linear system (4.6). By (3.4), we have the right preconditioned linear system

$$(4.7) \quad \left(\mathbf{I}_{N-1} + \text{diag}\{\mathbf{a}\} \tilde{\mathbf{D}}_{\mathbf{x} \rightarrow \mathbf{y}}^{(1)} \tilde{\mathbf{B}}_{\mathbf{y} \rightarrow \mathbf{x}}^{(-\nu)} + \text{diag}\{\mathbf{b}\} \tilde{\mathbf{D}}_{\mathbf{x} \rightarrow \mathbf{y}}^{(0)} \tilde{\mathbf{B}}_{\mathbf{y} \rightarrow \mathbf{x}}^{(-\nu)} \right) \mathbf{v} = \mathbf{f}.$$

It is easy to show that

$$\tilde{\mathbf{D}}_{\mathbf{x} \rightarrow \mathbf{y}}^{(1)} \tilde{\mathbf{B}}_{\mathbf{y} \rightarrow \mathbf{x}}^{(-\nu)} = \tilde{\mathbf{B}}_{\mathbf{y} \rightarrow \mathbf{y}}^{(1-\nu)}, \quad \tilde{\mathbf{D}}_{\mathbf{x} \rightarrow \mathbf{y}}^{(0)} \tilde{\mathbf{B}}_{\mathbf{y} \rightarrow \mathbf{x}}^{(-\nu)} = \tilde{\mathbf{B}}_{\mathbf{y} \rightarrow \mathbf{y}}^{(0-\nu)},$$

where

$$\tilde{\mathbf{B}}_{\mathbf{y} \rightarrow \mathbf{y}}^{(1-\nu)} = \left[\frac{d}{dx} (B_j^\nu(x)) \Big|_{x=y_i} \right]_{i,j=1}^{N-1}, \quad \tilde{\mathbf{B}}_{\mathbf{y} \rightarrow \mathbf{y}}^{(0-\nu)} = [B_j^\nu(y_i)]_{i,j=1}^{N-1}.$$

Then, the equation (4.7) reduces to

$$(4.8) \quad \left(\mathbf{I}_{N-1} + \text{diag}\{\mathbf{a}\} \widetilde{\mathbf{B}}_{\mathbf{y} \mapsto \mathbf{y}}^{(1-\nu)} + \text{diag}\{\mathbf{b}\} \widetilde{\mathbf{B}}_{\mathbf{y} \mapsto \mathbf{y}}^{(0-\nu)} \right) \mathbf{v} = \mathbf{f}.$$

After solving (4.8), we obtain \mathbf{u} by $\mathbf{u} = \widetilde{\mathbf{B}}_{\mathbf{y} \mapsto \mathbf{x}}^{(-\nu)} \mathbf{v}$.

Example 2. We consider the fractional differential equation (4.5) with

$$\nu = 1.9, \quad a(x) = 2 + \sin(4\pi x), \quad b(x) = 2 + \cos x.$$

The function $f(x)$ is chosen such that the exact solution of (4.5) is

$$u(x) = e^{x+1} - x - 2 - \frac{e^2 - 3}{4}(x+1)^2 + (x+1)^{46/7} - 2(x+1)^{39/7}.$$

Let $\{x_j\}_{j=0}^N$ be the Chebyshev points of the second kind (also known as Gauss-Chebyshev-Lobatto points) defined as

$$x_j = -\cos \frac{j\pi}{N}, \quad j = 0, 1, \dots, N,$$

and $\{y_j\}_{j=1}^{N-1}$ be the Gauss-Jacobi points as in Remark 3.4. We compare condition numbers, number of iterations (using BiCGSTAB in Matlab with TOL= 10^{-11}) and maximum point-wise errors of FSC and PFSC (see Figure 2). Observe from Figure 2 (left) that the condition number of FSC behaves like $\mathcal{O}(N^{3.8})$, while that of PFSC scheme remains a constant even for N up to 1024. As a result, PFSC scheme only requires about 13 iterations to converge (see Figure 2 (middle)), while the FSC scheme fails to converge (when $N \geq 16$) within N iterations as depicted in Figure 2 (right).

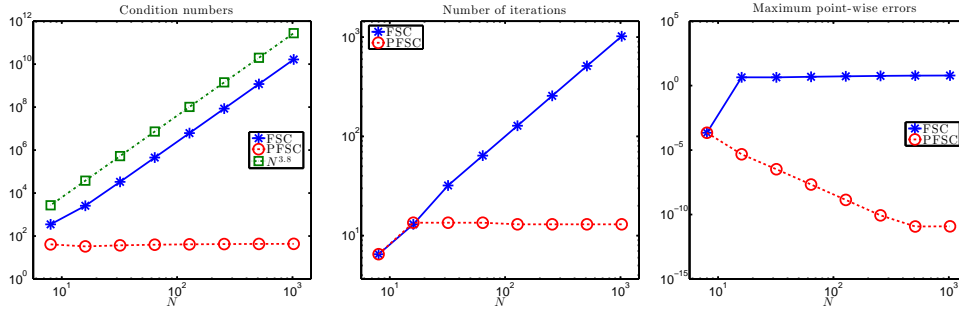


FIG. 2. Comparison of condition numbers (left), number of iterations (middle), and maximum point-wise errors (right) for Example 2.

5. Concluding remarks. We numerically show that the Birkhoff interpolation preconditioning techniques in [5, 3] are still effective for fractional spectral collocation schemes [7, 8, 2] based on fractional Lagrange interpolation. The preconditioned coefficient matrix is a perturbation of the identity matrix. The condition number is independent of the number of collocation points. The preconditioned linear system can be solved by an iterative solver within a few iterations. The application of the preconditioning FSC scheme to multi-term fractional differential equations is straightforward.

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